CHAPTER

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Normed Spaces

Norm # Let V be a vector space over the field F, where F is real or complex. A norm for V is a function $\|\cdot\|: V \longrightarrow R$ which satisfies the following

N_1: $\|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$

N_2: $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall \alpha \in F, \, x \in V$

N_3: $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$ Triangular property.

The pair $(V, \|\cdot\|)$ or simply V is called a normed linear space or normed vector space

Comments # $\|x\|$ is called the norm of x. Geometrically, the norm of x is the length of the vector x. The elements of the field is called scalars.

Example Show that R is a normed space over the field of real numbers.

Sol # We know that R is a vector space over itself and normed function is defined as

$$\|\cdot\| : R \rightarrow R$$

$$\|x\| = |x|$$

and it satisfies the properties of norm

N_1: $\|x\| = |x| \geq 0$ & $\|x\| = |x| = 0 \Leftrightarrow x = 0$

N_2: $\|\alpha x\| = |\alpha| \, |x| = |\alpha| \, \|x\|$

N_3: $\|x+y\| = |x+y| \leq |x| + |y| = \|x\| + \|y\|$

Hence R is a normed linear space or simply normed space.

Metric Defined By Norm

Every normed linear space N is a metric space with the metric $d: N \times N \to R$ defined by

$$d(x,y) = \|x-y\|$$

This metric is called the metric induced by the norm.

We show that d is a metric on N.

M_1: $d(x,y) = \|x-y\| \geq 0$

$d(x,y) = 0$

$\Leftrightarrow \|x-y\| = 0$

$\Leftrightarrow x-y = 0$

$\Leftrightarrow x = y$

M_2: $d(x,y) = \|x-y\| = \|-1(y-x)\|$

$= |-1| \, \|y-x\|$

$= \|y-x\|$

$= d(y,x)$

M_3: $d(x,y) = \|x-y\|$

$= \|x-z+z-y\|$

$\leq \|x-z\| + \|z-y\|$

$= d(x,z) + d(z,y)$

Hence d is metric

Q# Prove that $\left| \|x\| - \|y\| \right| \leq \|x-y\|$

Sol: we have

$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$

$\|x\| - \|y\| \leq \|x-y\| \longrightarrow (1)$

$$\|y\| = \|y - x + x\|$$
$$\leq \|y - x\| + \|x\|$$

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$$

$$-[\,\|x\| - \|y\|\,] \leq \|x - y\|$$

$$\|x\| - \|y\| \geq -\|x - y\| \longrightarrow ②$$

By ① & ②

$$|\,\|x\| - \|y\|\,| \leq \|x - y\|$$

Uniformty of Norm

The norm function $\|\cdot\| : N \rightarrow R$ is uniformly continuous.

Here for any $\epsilon > 0$, takig $\delta = \epsilon$. Then for all $x, y \in N$

$$d(x, y) = \|x - y\| < \delta \Rightarrow |\,\|x\| - \|y\|\,| \leq \|x - y\| < \epsilon$$

Theorem # For a normed linear space N.

(a) The function $f : N \times N \rightarrow N$ defined by

$$f(x, y) = x + y \qquad x, y \in N$$

is uniformly Continuous

(b) The function $g : F \times N \rightarrow N$ defined by

$$g(a, x) = ax \qquad a \in F, x \in N$$

is continuous

(c) For a fixed scalar a, the function $h_a : N \rightarrow N$ defined by $h_a(x) = ax$ is Continuous

Proof # (a) Given any $\epsilon > 0$ and $x, y \in N$ take $\delta = \epsilon/2$. Then δ dependups on ϵ only. for $x', y' \in N$,

$$\|f(x, y) - f(x', y')\| = \|x + y - x' - y'\|$$
$$= \|x - x' + y - y'\|$$

$$\leq \|x-x'\| + \|y-y'\|$$
$$< \epsilon/2 + \epsilon/2 < \epsilon$$

Hence f is uniformly continuous.

(b): Let $(a_0, x_0) \in F \times N$

$$\|g(a,x) - g(a_0, x_0)\|$$
$$= \|ax - a_0x_0\|$$
$$= \|(a-a_0)(x-x_0) + a_0(x-x_0) + x_0(a-a_0)\|$$
$$\leq |a-a_0|\|x-x_0\| + |a_0|\|x-x_0\| + |a-a_0|\|x_0\|$$
$$\to 0 \quad \text{as} \quad a \to a_0,\ x \to x_0$$

Hence $g(a,x) \to g(a_0, x_0)$ and so g is continuous.

(c) # For a fixed scalar a let

$$h_a(x) = ax \quad x \in N$$

Then if $a = 0$ $\quad h_a(x) = 0 \quad \forall x \in N$

Hence for every $\epsilon > 0$ and $x' \in N$ with $d(x, x')$ having any value

$$\|h_a(x) - h_a(x')\| = 0 < \epsilon$$

if $a \neq 0$; then choose $\delta = \epsilon/|a|$ so that

$$d(x,x') < \delta \Rightarrow \|ax - ax'\| = |a|\,\|x - x'\| < \epsilon$$

Hence h_a is uniformly continuous.

Theorem # (a) For x, y, z in a normed space N

$$d(x+z, y+z) = d(x,y)$$

(b) For all $a \in F$, $x, y \in N$

$$d(ax, ay) = |a|\, d(x,y)$$

Proof # (a) For any $x, y, z \in N$

$$d(x+z, y+z) = \|x+z-y-z\|$$
$$= \|x-y\| = d(x,y)$$

(b) For any scalar a & $x, y \in N$

$$d(ax, ay) = \|ax - ay\| = \|a(x-y)\| = |a|\|x-y\| = |a|\, d(x,y)$$

1 # **The Space R^n & C^n** #

The space R^n is normed linear space under the norm defined by for $x = (x_1, x_2, \ldots x_n) \in R^n$

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2} \quad \rightarrow (1)$$

The space C^n is normed linear space under the norm defined as for $z = (z_1, z_2, \ldots, z_n)$

$$\|z\| = \sqrt{\sum_{i=1}^{n} |z_i|^2} \quad \rightarrow (2)$$

Here the condition N_1 & N_2 are easy to varify.
For N_3

$$x = (x_1, x_2, \ldots x_n) \quad y = (y_1, y_2, \ldots y_n)$$

$$\|x+y\| = \sqrt{\sum_{i=1}^{n} |x_i+y_i|^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} |x_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i|^2} \quad (\because \text{Minkowski's inequality})$$

$$\leq \|x\| + \|y\|$$

The space R^n with norm defined by (1) is called n-dimensional Eucledean space
The space C^n with norm defined by (2) is called unitary space.

2 #

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The Space l_p

The space l_p consists of sequences $x = \{x_n\}$ such that

$$\sum_{i=1}^{\infty} |x_i|^p < \infty$$

We define norm on l_p as

$$\|x\| = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$$

Clearly

$\|x\| \geq 0$

now

$$\|x\| = 0$$

$$\Leftrightarrow \sum_{i=1}^{\infty} |x_i|^p = 0$$

$$\Leftrightarrow |x_i|^p = 0 \quad \forall i$$

$$\Leftrightarrow |x_i| = 0 \quad \forall i$$

$$\Leftrightarrow x_i = 0 \quad \forall i$$

$$\Leftrightarrow x = \{0, 0, 0, 0 \ldots \ldots\}$$

N_2: $\|\alpha x\| = \sqrt[p]{\sum |\alpha x_i|^p}$

$$= |\alpha| \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$$

$$= |\alpha| \, \|x\|$$

$\underline{N_3}$: Let $x = \{x_n\} \in l_p$ $y = \{y_n\} \in l_p$

$$\|x + y\| = \sqrt[p]{\sum_{i=1}^{n} |x_i + y_i|^p}$$

$$\leq \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} + \sqrt[p]{\sum |y_i|^p}$$

$$= \|x\| + \|y\| \quad \text{(Minkowski's Inequality)}$$

$\|x + y\| \leq \|x\| + \|y\|$

3 The Space l_p^n # $(1 \leq p < \infty)$

The elements of l_p^n are n-tuples $x = (x_1, x_2, \ldots \ldots x_n)$ $x_i \in R$

The norm in l_p^n is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

Here also the N_3 property can be proved by using Minkowski's Inequality.

4 # The Space l^n

The elements of this space are elements of R^n or C^n with norm defined by

$$\|x\| = \sup_{i=1}^{n} \{|x_i|\}$$

The space is commonly denoted by l_∞^n and the symbol $\|x\|_\infty$ is occasionally used for the norm. The reason for this practice lies in the interesting fact

$$\|x\|_\infty = \lim \|x\|_p \qquad p \to \infty$$

Here for any $x \in l^n$, $\|x\| \geq 0$

$$\|x\| = 0$$

$$\Leftrightarrow \sup_{i=1}^{n}\{|x_i|\} = 0$$

$$\Leftrightarrow x_i = 0 \quad \forall i$$

$$\Leftrightarrow x = \underline{0}$$

N_1 is satisfied.

N_2:

$$\|ax\| = \sup_{i=1}^{n}\{|ax_i|\}$$

$$= |a| \sup_{i=1}^{n}\{|x_i|\}$$

$$= |a| \, \|x\|$$

$$\|\underline{x}+\underline{y}\| = \sup_{i=1}^{n}\{|x_i+y_i|\}$$
$$\leq \sup_{i=1}^{n}|x_i| + \sup_{i=1}^{n}|y_i|$$
$$\leq \|x\| + \|y\| \quad \forall\, \underline{x},\underline{y} \in l^n$$

5# **The Space l^∞, c and c_0 #** The space l^∞ consists of all bounded sequences with norm defined as follows

For $x = \{x_n\} \in l^\infty$, with $|x_n| < k$ for some fixed real number K, for all $n = 1, 2, \ldots$

$$\|x\| = \sup_{i=1}^{\infty} |x_i|$$

The l^∞ with this norm, is a normed linear space

The space c, c_0 are subspaces of l^∞. c consists of all const sequences in l^∞ while c_0 consists of all those sequences in l^∞ which converge to zero.

Both these spaces have the norm as defined in l^∞

6# **The Space $C[a\ b]$ #**

This space consists of all continuous real valued functions defined on $[a\ b]$

For $f \in C[a\ b]$

$$\|f\| = \sup_{x\in[a\ b]} |f(x)|$$

We define norm on C [a b] as follows
For $f \in C[a\ b]$, we put

$$\|f\| = \sup_{x \in [a\ b]} |f(x)|$$

Then C [a b], with this norm, is a normed linear space

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The Space B(X R)

This space consists of all bounded real valued functions defined on a arbitrary topological space X. This is a linear space with multiplication and addition defined by

$$(f+g)(x) = f(x) + g(x)$$

$$(af)(x) = a f(x)$$

The norm on B(X R) is defined by

$$\|f\| = \sup_{x \in X} |f(x)|$$

The Space C [a b]

It consists of all continuous functions defined on [a b]. norm on it is defined by

$$\|f\| = \int_a^b |f(x)|\, dx \qquad x \in [a\ b]$$

Here $\|f\| \geq 0$

$$\|f\| = 0$$
$$\Leftrightarrow f(x) = 0 \quad \forall x \in [a, b]$$
$$\Leftrightarrow f = 0$$

N_2: $\|\alpha f\| = |\alpha| \|f\|$

N_3:

$$\|f+g\| = \int_a^b |f(x)+g(x)|\,dx.$$
$$\le \int_a^b |f(x)|\,dx + \int_a^b |g(x)|\,dx$$
$$\le \|f\| + \|g\|$$

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Banach Space

A complete normed space is called a banach space.

Comments #1: A normed linear space N is com Banach space if N as a metric space with metric defined by norm of N is complete.

2# Many important properties of Banack space are closely linked to the shape of its closed unit sphere i.e $S = \{x : \|x\| \le 1\}$. One basic property of S is that it is always convex in the sense that if x, y are any two vectors in S, Then vector $z = \alpha x + \beta y$ is also in S, where α, β are non-negative real number such that $\alpha + \beta = 1$

Examples

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1) **The space R^n or C^n**

The spaces R^n & C^n are Banach spaces. We prove that R^n is complete

Let $\{x^{(p)}\}$ be a Cauchy sequence in R^n

$$x^{(p)} = (x_1^{(p)}, x_2^{(p)}, \ldots, x_n^{(p)})$$

Then given any $\epsilon > 0$, there is a natural no n_0 such that

$$\| x^{(p)} - x^{(q)} \| < \epsilon \qquad \forall\, p, q \geq n_0$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n} | x_i^{(p)} - x_i^{(q)} |^2} < \epsilon \qquad \forall\, p, q \geq n_0$$

$$| x_i^{(p)} - x_i^{(q)} | \leq \| x^{(p)} - x^{(q)} \| < \epsilon \quad \forall\, p, q \geq n_0$$

So for each i, $\{x_i^{(p)}\}$ is a Cauchy sequence of real numbers. Since R is complete, $\{x_i^{(p)}\}$ converges to a real number x_i say $i = 1, 2, \ldots$

Now for $\epsilon > 0$ already chosen, there are natural numbers n_i such that

$$| x_i^{(p)} - x_i | < \epsilon/n \qquad p \geq n_i \rightarrow (1)$$

Take $x = (x_1, x_2, \ldots, x_n)$

where $x_i = \lim_{p \to \infty} x_i^{(p)}$. Then $x \in R^n$

We show that $\lim_{p \to \infty} x^{(p)} = x$

Let $n = \max\{n_1, n_2, \ldots, n_n\}$

Then

$$\| x^{(p)} - x \| = \sqrt{\sum_{i=1}^{n} | x_i^{(p)} - x_i |^2}$$

$$< n\, \epsilon/n = \epsilon \qquad p \geq n$$

Hence $\{x^{(p)}\}$ converges to $x \in R^n$ as required.

Thus R^n is complete and hence a Banach space

The Space l^∞

This is space of all bounded sequences of real or complex numbers with addition and scalar multiplication defined by

$$x+y = \{x_i + y_i\}$$
$$\alpha x = \{\alpha x_i\}$$

The norm in l^∞ is defined by

$$\|x\| = \sup_{i=1}^{\infty} |x_i|$$

We show that l^∞ is a Banach space.

Let $\{x^{(p)}\}$ be any cauchy sequence in l^∞, $x^{(p)} = \{x_i^{(p)}\}$.

Then given $\epsilon > 0$, there is a natural number n_0 such that

$$\|x^{(p)} - x^{(q)}\| = \sup_{i=1}^{\infty} |x_i^{(p)} - x_i^{(q)}| < \epsilon \quad \forall\, p, q \geq n_0$$

So for each i

$$|x_i^{(p)} - x_i^{(q)}| \leq \|x^{(p)} - x^{(q)}\| < \epsilon \quad \forall\, p, q \geq n_0$$

Hence $\{x_i^{(p)}\}$ is a cauchy sequence of real numbers for each i. Since R is complete, $\{x_i^{(p)}\}$ converges to x_i say for each $i = 1, 2, \ldots$

Let $x = \{x_i\}$. We show that $x \in l^\infty$ and $\lim_{p \to \infty} x^{(p)} = x$

Since $x_i^{(p)} \longrightarrow x_i$, there is a natural number n_1 s.t

$$|x_i^{(p)} - x_i| < \epsilon/2 \quad i = 1, 2, \ldots \quad \forall\, p \geq n_1 \quad \rightarrow (1)$$

i.e

$$\sup_{i=1}^{n} |x_i^{(p)} - x_i| < \epsilon/2$$

$$\Rightarrow \|x^{(p)} - x\| < \epsilon/2 \quad \forall\, p \geq n_1$$

Hence $x^{(p)} \rightarrow x$.

Again from

$p \geq n_1 \Rightarrow |x_i^{(p)} - x_i| < \epsilon/2 \quad i = 1, 2$

$$|x_i| = |x_i - x_i^{(p)} + x_i^{(p)}|$$

$$\leq |x_i - x_i^{(p)}| + |x_i^{(p)}|$$

$$< \epsilon/2 + k_p \quad \because x_i^{(p)} \text{ is bounded.}$$

which is a finite number, independent of i. Hence $\{x_i\} \in l^\infty$

Thus l^∞ is Complete and hence a Banach space.

The Space C

This is the space of Convergent real (or Complex) sequences. Since a Cgt sequence is bounded, Therefore members of C are also members of l^∞. Hence C is a subspace of l^∞

We show that C is a Banach space by showing that C is a closed subspace of l^∞.

Let $\underline{x}$ be a limit point of C. Then $x \in l^\infty$ so that $\underline{x} = \{x_i\}$. By definition of limit point, there is a sequence $\{x^{(p)}\}$ in C such that

$$\lim_{p \to \infty} x^{(p)} = x$$

Hence given $\epsilon > 0$ there is a natural number n_0 such that

$$p \geq n_0 \Rightarrow \|x^{(p)} - x\| = \sup_{i=1}^{\infty} |x_i^{(p)} - x_i| < \epsilon/3$$

Consider now the sequence $x^{(n_0)} = \{x_i^{(n_0)}\}$

Then $x^{(n_0)} \in C$ and so is Convergent. But a Convergent sequence is cauchy. Hence for ϵ

already chosen, there is a natural number n_1 such that

$$|x_i^{(n_0)} - x_j^{(n_0)}| < \epsilon/3 \qquad \forall\, i,j \geq n_1$$

Let $n_2 = \max(n_0, n_1)$

Then.

$$|x_i - x_j| = |x_i - x_i^{(n_0)} + x_i^{(n_0)} - x_j^{(n_0)} + x_j^{(n_0)} - x_j|$$

$$\leq |x_i - x_i^{(n_0)}| + |x_i^{(n_0)} - x_j^{(n_0)}| + |x_j^{(n_0)} - x_j| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \quad i,j \geq n_2$$

Thus $\{x_i\}$ is a cauchy sequence in R (or C). Since R (or C) is complete, $\{x_i\} = x$ is convergent and hence is in C. Thus C is closed subspace of l^∞. But then C is complete and hence is a Banach space.

Theorem # A subspace "A" of a Banach space X is complete iff it is closed in X

Proof # Suppose 'A' is a subspace of a Banach space X and is complete.

we show that A is closed in X i.e 'A' contains all of its limit points.

Let x be a limit point of A. Then there is a sequence $\{x_n\}$ in 'A' such that

$$x_n \longrightarrow x$$

Since every convergent sequence is cauchy sequence, therefore $\{x_n\}$ is a cauchy sequence in X. Since 'A' is complete, $\{x_n\}$ converges to a point of A. So $x \in A$. Hence A is closed.

Conversely suppose that A is closed subspace of X we show that A is complete

Let $\{x_n\}$ be a cauchy sequence in A. Then $\{x_n\}$ is also a cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to a point x of X. But then x is a limit point $\{x_1, x_2, x_3 \dots x_n \dots\} \in A$. Since A is closed, $x \in A$. Therefore every cauchy sequence in A converges in A. Hence A is complete.

Cor # Every closed subspace of a Banach space is itself a Banach space.

Proof # Let X be a Banach space and A be closed subspace of X. We show that 'A' is a Banach space.

Since 'A' is a subspace of Banach space i.e a complete normed space. Therefore 'A' itself is a normed space under the restriction of norm defined on X.

Further let $\{x_n\}$ be a cauchy sequence in "A", then $\{x_n\}$ is also a cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to a point x of X. But then 'x' is a limit point of $\{x_1, x_2, \dots x_n \dots\}$. Since "A" is closed, therefore $x \in A$

Therefore "A" is complete.

Th Space C_o

The space C_o consists of all sequences converging to zero. This is also a Banach space. The proof of this is same as in the above example.

The Space $C[a, b]$

This is the space

of all continuous functions from $[a, b]$ to R or C.
The norm in $C[a, b]$ is defined by

$$\|f\| = \sup_{x \in [a, b]} |f(x)|, \quad f \in C[a, b]$$

Let $\{f^{(m)}\}$ be a Cauchy sequence in $C[a, b]$.
Then given $\epsilon > 0$, there is a natural number n_0 such that

$$\|f^{(m)} - f^{(n)}\| < \epsilon/3 \qquad \forall\, m, n \geq n_0 \quad \rightarrow (1)$$

$$\Rightarrow \sup_{x \in [a, b]} |f^{(m)}(x) - f^{(n)}(x)| < \epsilon/3$$

By Cauchy criterion for uniform convergence of a sequence of function $\{f^{(m)}\}$ converges to a function (f). Since each $f^{(m)}$ is continuous, f as a uniform limit of a sequence of continuous functions, is continuous on $[a, b]$. Thus $f \in C[a, b]$ so that $C[a, b]$ is complete.

Problem # Show that c_0 (the space of all sequences converging to zero) is closed subspace of l^∞, so that c_0 is complete (or Banach space).

Sol # We show that c_0 is closed by showing

$$\overline{c_0} = c_0$$

Clearly by definition

$$c_0 \subseteq \overline{c_0} \longrightarrow ①$$

To show that $\overline{c_0} \subseteq c_0$

Let $x = \{x_i\} \in \overline{c_0}$

Then there is a sequence $x^{(n)} = \{x_i^{(n)}\}$ in c_0 such that

$$x^{(n)} \longrightarrow x \text{ as } n \longrightarrow \infty$$

Then for a given $\epsilon > 0$ there is a +ve number n_0 such that

$$\| x^{(n)} - x \| < \epsilon/2 \qquad \forall\, n \geq n_0$$

Let us consider a particular case in which $n = n_0$

Since $x^{(n_0)} \in c_0$

i.e. $x_i^{(n_0)} \longrightarrow 0$

$\because$ for every $\epsilon > 0$ $\exists$ a natural number n_1 such that

$$| x_i^{(n_0)} - 0 | = | x_i^{(n_0)} | < \epsilon/2 \qquad n_0 \geq n_1$$

$$|x_i| = |x_i - x_i^{(n_0)} + x_i^{(n_0)}|$$

$$\leq |x_i - x_i^{(n_0)}| + |x_i^{(n_0)}|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \qquad \forall\, \text{~~n ≥ n₂ = m(.~~}$$

$\Rightarrow x_i \longrightarrow 0$

Thus $x = \{x_i\} \in c_0$

$\Rightarrow x \in \overline{c_0}$

Hence $\overline{c_0} = c_0$

$\Rightarrow c_0$ is closed.

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Basis of a Normed Space

A Collection $B = \{e_1, e_2, \ldots e_n\}$ of elements of a normed space N is said to be a basis for N if

(a) The set B is linearly independent

(b) For each $x \in N$, there is a unique representation

$$x = a_1e_1 + a_2e_2 + \ldots + a_ne_n.$$

$$= \sum_{i=1}^{n} a_ie_i.$$

i.e

$$\lim_{n \to \infty} \|x - \sum_{i=1}^{n} a_ie_i\| = 0$$

Finite Dimensional Normed Space

A normed space N is said to be finite dimensional if it has a finite basis. Otherwise N is said to be infinite dimensional

Lemma # Let $\{x_1, x_2 \ldots x_n\}$ be a linearly independent set of vectors in a normed space X. Then there is a real no $c > 0$ such that for every choice of scalars $\alpha_1, \alpha_2 \ldots \alpha_n$, we have

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq c(|\alpha_1| + \ldots + |\alpha_n|)$$

Proof # Let $s = |\alpha_1| + |\alpha_2| + |\alpha_3| + \ldots + |\alpha_n|$

There are two cases

Case-I # If $s = 0$

Then the relation

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq c(|\alpha_1| + \ldots + |\alpha_n|)$$

holds, because $s = 0$

$\Rightarrow |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| = 0$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0 \cdots\cdots, \alpha_n = 0$

This is trivial

Case – II # If $s \neq 0$

; then dividing both sides of

$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n\| \geq c\,(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|)$

by s we have

$$\|\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n\| \geq c \qquad \left(\sum_{i=1}^{n} |\beta_i| = 1\right) \longrightarrow ①$$

where $\beta_i = \dfrac{\alpha_i}{s}$ $\quad i = 1, 2, \ldots n$

It is sufficient to prove the existence of a no $c > 0$ such that ① holds for every n-tuples of scalars $\beta_1, \beta_2 \cdots \beta_n$ with $\sum_{n=1}^{n} |\beta_n| = 1$

Suppose this is not True. Then there exists a sequence $\{y_m\}$ of vectors

$$y_m = \beta_1^m x_1 + \beta_2^m x_2 + \cdots \beta_n^m x_n$$

such that

$$\|y_m\| \to 0 \quad \text{as } m \to \infty$$

It is so because when ① does not hold we have for a sequence of scalars $\{\beta_i^{(m)}\}$ that

$$\|\beta_1^m x_1 + \beta_2^{(m)} x_2 + \cdots \beta_n^{(m)} x_n\| < c$$

Since $\displaystyle\sum_{i=1}^{n} |\beta_i^m| = 1$

Therefore $|\beta_i^m| \leq 1$

Hence, for each fixed i the sequence $\{\beta_i^m\} = \{\beta_i^1, \beta_i^2 \cdots\}$ is bounded.

Consequently by Bolzano Weirstrass Theorem $\{\beta_1^m\}$ has a convergent subsequence. Let β_1 denote the limit of that subsequence and

$\{y_{1,m}\}$ denote the Corresponding Sub-sequence of $\{y_m\}$.

By the same argument $\{y_m\}$ has a subsequence has a Subsequence $\{y_{2,m}\}$, for which the Corresponding subsequence of scalars β_2^m Converges. Let β_2 denote the limit.

Continuing in the same way after n-steps, we obtain a subsequence $\{y_{n,m}\} = \{y_{n,1}, y_{n,2} \ldots\}$ of y_m which has terms

$$y_{n,m} = \sum_{i=1}^{n} \gamma_i^m x_i \qquad \left(\sum_{i=1}^{n} |\gamma_i^m| = 1 \right)$$

Satisfying $\gamma_i^m \longrightarrow \beta_i$ as $m \to \infty$ since γ_i^m is sub-sequence of β_i^m and $\beta_i^m \to \beta_i$

Hence as $m \to \infty$

$$y_{n,m} \longrightarrow y = \sum_{i=1}^{n} \beta_i x_i$$

where $\sum_{i=1}^{n} |\beta_i| = 1$

Clearly not all β_i are zero and since $\{x_1, x_2 \ldots, x_n\}$ is a linearly independent set so $y \neq 0$ ($\because$ If $y = 0$, then all β_i will be zero, but we have proved that not all β_i are zero)

On the other hand by the continuity of norms

$$y_{n,m} \longrightarrow y$$

$$\Rightarrow \|y_{n,m}\| \longrightarrow \|y\| \longrightarrow (2)$$

Since $\|y_m\| \to 0$ (By assumption)

and $\{y_{n,m}\}$ is a subsequence of $\{y_m\}$

$\therefore$ $\|y_{n,m}\| \to 0$ so $\|y\| = 0$ from (1)

$\Rightarrow y = 0$ This is a contradiction.

Hence

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq c \left(|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|\right)$$

Theorem # Every finite dimensional subspace of a normed space X is complete. In particular every finite dimensional normed space is Complete (i.e a Banach Space)

Proof # Let A be a finite dimensional subspace of normed linear space X. Let $\{e_1, e_2, \dots e_n\}$ be a basis of A. Let $\{x_n\}$ be a cauchy sequence in A. Then for a given $\epsilon > 0$, $\exists$ a natural number N s.t

$$\|x_m - x_p\| < \epsilon \qquad \forall\, m, p \geq N$$

Now

$$x_m = \sum_{i=1}^{n} \overset{m}{\alpha_i} e_i$$

or

$$x_m = \overset{m}{\alpha_1} e_1 + \overset{m}{\alpha_2} e_2 + \dots + \overset{m}{\alpha_n} e_n$$

$$x_p = \overset{p}{\alpha_1} e_1 + \overset{p}{\alpha_2} e_2 + \dots + \overset{p}{\alpha_n} e_n$$

We know that if $\{x_1, x_2 \dots x_n\}$ be a linearly independent set of vectors in a normed space X, then there is a real no $c > 0$ for every choice of scalars $\alpha_1, \alpha_2 \dots \alpha_n$, we have

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\| \geq c(|\alpha_1| + \dots + |\alpha_n|)$$

Using this result

$$\epsilon > \|x_m - x_p\| = \|\sum \|(\overset{m}{\alpha_i} - \overset{p}{\alpha_i}) e_i\|\| \geq c\, |\overset{m}{\alpha_i} - \overset{p}{\alpha_i}|$$

$$\Rightarrow \sum_{i=1}^{n} |\overset{m}{\alpha_i} - \overset{p}{\alpha_i}| < \epsilon/c \qquad \forall\, m, p \geq N \longrightarrow (1)$$

In particular $|\overset{m}{\alpha_i} - \overset{p}{\alpha_i}| < \epsilon/c \quad \forall\, m, p \geq N$

So $\{\alpha_i^m\}$ is a Cauchy sequence in R or C. Since R & C are complete, Therefore

$\{\alpha_i^m\}$ Converges to $\alpha_i \in R$ or C for each i

Put $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$ $x \in A$
we show that $\{x_m\} \longrightarrow x$
But then

$$\|x_m - x\| = \| \sum (\alpha_i^m - \alpha_i) e_i \|$$

$$\leq k \sum_{i=1}^{n} |(\alpha_i^m - \alpha_i)|$$

where $k = \max_{i=1}^{n} \|e_i\|$

$\Rightarrow \|x_m - x\| < \frac{k\epsilon}{c}$ $\forall\, m \geq N$ by ①

Hence $\lim_{m\to\infty} x_m = x \in A$

Therefore A is Complete i.e Banach space

Note # The statements of the above Theorem can also be given by
(1) # Every finite dimensional normed space is a Banach space
(2) # Every finite dimensional subspace of a normed space is complete.

Corollary # Every finite dimensional subspace of a normed linear space X is closed in X.
Proof # Let A be a finite dimensional

of a normed linear space X. Then A is complete by above Theorem. We know that every finite dimensional subspace of a normed linear space X is complete. Also by a previous Theorem we know that A subspace A of a complete space X is complete iff it is closed in X
∴ Every finite dimensional subspace of a normed linear space X is closed in X

Remark # An infinite dimensional subspace in a normed space may not be closed.

Example # Let $X = C[0\ 1]$, the space of continuous functions on the interval $0 \le x \le 1$ and $S = \text{Span}\{u_0, u_1, \dots\}$, where $u_i(x) = x^i \quad i = 0, 1, 2 \dots$ This means that S is a subspace of polynomials. It is clear that S is not closed.

Remarks # We know that a normed space may finite or infinite dimensional. We can associate different norms. The question arise whether these norms are equivalent or not. The answer to this question is partly true. If the normed space is finite dimensional, then all the norms defined on it are equivalent. However if the normed space is infinite dimensional, then these norms on the space may not be equivalent.

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Equivalent Norms

Let X be a normed linear space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be equivalent if there are non-zero real numbers "a" and (b) such that

$$a\|x\|_2 \le \|x\|_1 \le b\|x\|_2 \rightarrow (1)$$

$\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ and is written as
$\|\cdot\|_1 \sim \|\cdot\|_2$

Theorem# Any two norms on a finite dimensional linear space are equivalent.

Proof# Let X be a finite dimensional normed linear space.

Let $\|\cdot\|_1$ & $\|\cdot\|_2$ be any two norms defined on X. We show that $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent.

Let $\{e_1, e_2, \ldots, e_n\}$ b a basis of X and $x \in X$. Then

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$$

$$\|x\|_1 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|$$

$$\ge c\left(|\alpha_1| + \cdots + |\alpha_n|\right) \quad c > 0$$

$$\Rightarrow \|x\|_1 \ge c \sum_{i=1}^{n} |\alpha_i| \quad , \quad c > 0$$

$$\|x\|_1 \ge c s \quad , \quad \text{where } s = \sum_{i=1}^{n} |\alpha_i|$$

Also $\|x\|_2 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|$

$$\le |\alpha_1|\,\|e_1\| + \cdots + |\alpha_n|\,\|e_n\|$$

$$\le k \sum_{i=1}^{n} |\alpha_i| \qquad k = \max_{i=1}^{n} \|e_i\|_2$$

$$= ks \rightarrow (2)$$

from (1) & (2)

$$\frac{1}{k}\|x\|_2 \le s \le \frac{1}{c}\|x\|_1$$

or $$\frac{c}{k}\|x\|_2 \le \|x\|_1$$

$$a\|x\|_2 \leq \|x\|_2$$

Again $\|x\|_1 = \|\sum_{i=1}^{n} \alpha_i e_i\|$

$$\leq \sum_{i=1}^{n} |\alpha_i| \|e_i\|$$

Let $k' = \max_{i=1}^{n} \{\|e_i\|\}$

Then

$$\|x\|_1 \leq k' s \qquad \text{where } s = \sum_{i=1}^{n} |\alpha_i| \rightarrow ③$$

and $\|x\|_2 = \|\sum_{i=1}^{n} \alpha_i e_i\| \geq c' \sum_{i=1}^{n} |\alpha_i|$

$$\|x\|_2 \geq c' s \qquad s = \sum_{i=1}^{n} |\alpha_i| \rightarrow ④$$

from ③ & ④

$$\|x\|_1 \leq \frac{k'}{c'} \|x\|_2$$

or $\|x\|_1 \leq b\|x\|_2 \qquad b = \frac{k'}{c'} \rightarrow$ Ⓑ

from Ⓐ & Ⓑ

$$a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2$$

Hence any two norms a finite dimensional normed space are equivalent.

Theorem # The relation of " being equivalent to " betweeen norms defined on a normed space is an equivalent relation.

Proof # For any norm on a space N the condition

$$a\|x\| \leq \|x\| \leq b\|x\| \quad \text{for } x \in N$$

and for $a = b = 1$

So the relation of being equivalent to is Reflexive.

Next if $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$, then there are real numbers a & b such that

$$a\|x\|_2 \le \|x\|_1 \le b\|x\|_2$$

But then

$$\|x\|_2 \le \frac{1}{a}\|x\|_1 \quad \& \quad \frac{1}{b}\|x\|_1 \le \|x\|_2$$

So $\frac{1}{b}\|x\|_1 \le \|x\|_2 \le \frac{1}{a}\|x\|_1$

Hence $\|\cdot\|_2 \sim \|\ \|_1$

$\Rightarrow$ the relation is symmetric.

Lastly if $\|\cdot\|_1 \sim \|\cdot\|_2$ & $\|\ \|_2 \sim \|\cdot\|_3$, then there are non-zero +ve real real numbers a, b & c, d such that

$$a\|x\|_2 \le \|x\|_1 \le b\|x\|_2 \rightarrow (1)$$

$$c\|x\|_3 \le \|x\|_2 \le d\|x\|_3 \rightarrow (2)$$

By (ii) by multiplying by a

$$ac\|x\|_3 \le a\|x\|_2 \le ad\|x\|_3$$

But

$$a\|x\|_2 \le \|x\|_1 \le b\|x\|_2 \quad \text{by } (1)$$

and

$$bc\|x\|_3 \le b\|x\|_2 \le bd\|x\|_3 \text{ from } (2)$$

Combining these relations

$$ac\|x\|_3 \le a\|x\|_2 \le \|x\|_1 \le b\|x\|_2 \le bd\|x\|_3$$

Hence

$$ac\|x\|_3 \le \|x\|_1 \le bd\|x\|_3$$

$\Rightarrow \|\cdot\|_1 \sim \|\cdot\|_3$

Thus this relation is an equivalence relation

Theorem # Any two ~~no~~ equivalent norms on a linear normed space N define the same topology

Proof # Let N be a normed linear

space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms defined on N. Then there are real numbers a and b such that

$$a\|x\|_2 \le \|x\|_1 \le b\|x\|_2 \quad \forall x \in N$$

Let $\mathcal{T}_1$ & $\mathcal{T}_2$ be topologies defined by the norms $\|\cdot\|_1$ and $\|\cdot\|_2$.

Let $U \in \mathcal{T}_1$ be an open set in $(N\ \mathcal{T}_1)$. Then for each $x \in U$, there is an open ball $B_r(x)$ such that

$$x \in B_r(x) \subseteq U$$

Now $B_r(x) = \{ y \in X : d(x\ y) = \|x-y\|_1 < r \}$

Consider now, an open ball $B'_{r'}(x)$ in $(N \ \|\ \|_2)$, where $r' = r/b$. Then for any $y \in B'_{r'}(x)$, we have

$$d_1(x,y) = \|x-y\|_1 \le b\|x-y\|_2$$
$$< b \cdot r/b = r$$

$$\Rightarrow y \in B_r(x)$$

$$\Rightarrow B'_{r'}(x) \subseteq B_r(x)$$

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Hence $B_r(x)$ is open in $(N\ \|\cdot\|_2)$. Similarly every open ball in $(N, \|\ \|_2)$ is open in $(N, \|\ \|_1)$. Hence the topologies induced by norms $\|\cdot\|_1$ & $\|\cdot\|_2$ on these two spaces are the same.

Theorem # If $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent norms defined on a normed space X, then every cauchy sequence with respect to one norm is also a cauchy sequence w.r.t the other norm.

Proof # Let X has two equivalent norms $\|\cdot\|_1$ & $\|\cdot\|_2$ defined on it.

there exist real numbers a & b such that

$$a\|x\|_2 \le \|x\|_1 \le b\|x\|_2 \quad \rightarrow (1)$$

Let $\{x_n\}$ be a cauchy sequence w.r.t $\|\cdot\|_1$, then given any $\epsilon > 0$, there is a natural number n_0 s.that

$$\|x_m - x_n\|_1 < \epsilon \quad \forall\, m, n \geq n_0 \quad \rightarrow (2)$$

But then

$$\|x_m - x_n\|_2 \le \frac{1}{a}\|x_m - x_n\|_1 < \epsilon/a \quad \forall\, m,n \geq n_0$$

by (1)

Hence $\{x_n\}$ is cauchy sequence w.r.t $\|\ \|_2$

Similarly if $\{x_n'\}$ is a cauchy sequence w.r.t $\|\ \|_2$, then given $\epsilon > 0$, there is a natural no n_0 s.t

$$m, n \geq n_0 \Rightarrow \|x_m' - x_n'\|_2 < \epsilon \quad \rightarrow (3)$$

So

$$m, n \geq n_0 \Rightarrow \|x_m' - x_n'\|_1 \le b\|x_m' - x_n'\|_2 < b\epsilon$$

Hence $\{x_n'\}$ is a cauchy sequence w.r.t $\|\cdot\|_1$ as well

<u>Corollary</u># If $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent norms on a linear space N, then sequence $\{x_n\}$ in $(N\ \|\cdot\|_1)$ converges to $x \in N$ iff $\{x_n\}$ converges to x in $(N\ \|\ \|_2)$

<u>Proof</u> Suppose $\{x_n\}$ converges to x in $(N\ \|\cdot\|_1)$
Then given $\epsilon > 0$ $\exists$ natural no n_0 such that

$$\|x_n - x\|_1 < \epsilon \quad \forall\, n \geq n_0$$

$$n \geq n_0 \quad \|x_n - x\|_2 \le \frac{1}{a}\|x_n - x\| < \epsilon/a \quad \text{by } (1)$$

Hence $\{x_n\}$ converges to x in $(N,\ \|\ \|_2)$

Theorem # Let $\|\cdot\|_0$ and $\|\cdot\|$ be equivalent norms on a normed space N_1, where $\|\cdot\|_0$ is defined by

$$\|x\|_0 = \sup_{i=1}^{\infty} |x_i|$$

Let N be finite dimensional subspace of N_1, then

Proof # Since $\|\cdot\|_0$ & $\|\cdot\|$ are equivalent, therefore there exist +ve real numbers a & b such that

$$a\|x\|_0 \le \|x\| \le b\|x\|_0$$

Also since N is finite dimensional, N has a finite basis

Let $\{e_1, e_2, \dots e_n\}$ be a basis of N, then for each $x \in N$

$$x = \sum_{i=1}^{n} x_i e_i$$

To show that N is complete, let $\{x^{(p)}\}$ be a cauchy sequence in N. Then

$$x^{(p)} = \sum_{i=1}^{n} x_i^{(p)} e_i$$

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Also for $\epsilon > 0$, there is a natural number n_0 such that $p, q \ge n_0$

$$\Rightarrow \|x^{(p)} - x^{(q)}\| < \epsilon$$

Now $$x^{(p)} - x^{(q)} = \sum_{i=1}^{n} (x_i^{(p)} - x_i^{(q)}) e_i$$

So $$\|x^{(p)} - x^{(q)}\|_0 = \sup |x_i^{(p)} - x_i^{(q)}|$$

$$\|x^{(p)} - x^{(q)}\| \ge a\|x^{(p)} - x^{(q)}\|_0 = a \sup_{i=1}^{n} |x_i^{(p)} - x_i^{(q)}| \quad \forall\, p, q \ge n_0$$

So

$p, q \geq n_0 \Rightarrow |x_i^{(p)} - x_i^{(q)}| < \epsilon/a$

Hence $\{x_i^{(p)}\}$ is a cauchy sequence in R. Since R is complete, $\{x_i^{(p)}\}$ converges to a point x_i of R.

i.e $|x_i^{(p)} - x_i| \to 0$ as $p \to \infty$

Now define $x \in N$ by

$$x = \sum_{i=1}^{n} x_i e_i$$

So $\|x^{(p)} - x\| \leq b\|x^{(p)} - x\|_0 = b \sup_{i=1} |x_i^{(p)} - x_i| \to 0$ as $p \to \infty$

$\Rightarrow \|x^{(p)} - x\| \to 0$ as $p \to \infty$

Hence

$$\lim_{p \to \infty} x^{(p)} = x \in N$$

$\Rightarrow N$ is complete

<u>Note</u> An infinite dimensional subspace of N may not be complete. For example

Take $N = C[a\ b]$

where $C[a.b]$ is the space of all continuous functions defined on $[a\ b]$

An element of $C[a\ b]$ is a function $x : [a\ b] \to R$ so that

$$x(t) \in R \quad \forall t \in [a.b]$$

By the Stone Weirestrass Theorem

" every function in $C[a\ b]$ can be approximated by polynomials of the form

$$\phi(x) = a_0 + a_1 x + \cdots + a_n x^n$$

i.e given any $f \in [a\ b]$, there is a sequence of polynomials $g^{n}(x)$ such that

$$f(t) = \lim_{n \to \infty} g^{(n)}(t)$$

Consider the functions

$1, x, x^2, \dots x^n, \dots$

which are polynomials and let P be the space generated by these polynomials. These polynomials are linearly independent, so they generate an infinite dimensional subspace of $C[a\ b]$

Then P is not closed in $C[a.b]$, because here each $f \in C[a\ b)$ is a limit pt. of P However not every function in $C[a\ b]$ is polynomial

Hence P is not closed i.e $\bar{P} \supset P$ and of course $\bar{P} = C[a\ b]$

Therefore $P \subset \bar{P} = C[a\ b]$

so that P is not closed in $C[a\ b]$ and hence is not complete.

Problem # Prove that the norms on n-dimensional Euclidean space R^n are equivalent

1)# $\|x\|_0 = \sup_{i=1}^{n} |x_i| \qquad x \in R^n$

2)# $\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad x \in R^n$

Sol # Let $x = (x_1, x_2, \dots x_n) \in R^n$. Then for any $y = (y_1, y_2, \dots, y_n)$ in R^n, we have

$$\left| \sum_{i=1}^{n} x_i y_i \right| \le \sum_{i=1}^{n} |x_i y_i| = \sum_{i=1}^{n} |x_i| \, |y_i|$$

$$\leq \sup_{i=1}^{n} |x_i| \sum_{i=1}^{n} |y_i|$$

$$\leq \|x\|_0 \, \|y\|_1$$

Taking $y = (1, 1, \ldots, 1)$, Then

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |x_i y_i|$$

$$\leq \sup_{i=1}^{n} |x_i| \sum_{i=1}^{n} |y_i|$$

$$\leq n \|x\|_0$$

Also

$$\|x\|_0 = \sup_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} |x_i| = \|x\|_1$$

Hence

$$\|x\|_0 \leq \|x\|_1 \leq n \|x\|_0$$

Thus $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent.

Compact Normed Spaces

There are two types of compactness of topological spaces namely compactness and sequential compactness. In general topological space these two concepts are different But in metric spaces these coincide. Thus when we talk about the compactness of normed spaces it would be in the sense of sequential compactness.

Compact Normed Space

A normed space $(N, \|\cdot\|)$ is compact iff every sequence in N has a cgt subsequence

Compact Subset

A subset of a normed space is compact if it is compact as a topological subspace of N.

Lemma A compact subset M of a metric space (X, d) is closed and bounded.

Proof Let $\overline{M}$ denote the closure of compact subset M of (X, d).

Let $x \in \overline{M}$, then there is a sequence $\{x_n\}$ in M such that

$$x_n \longrightarrow x$$

Since M is compact and $\{x_n\}$ is a sequence in M which is convergent, so $x \in M$

$\therefore \quad \overline{M} \subseteq M \rightarrow (1)$

But $M \subseteq \overline{M} \rightarrow (2)$

$\Rightarrow \quad M = \bar{M}$

Hence M is closed

Next to show that M is bounded. Suppose M is not bounded.

Then for a fixed $b \in M$ and any integer "n" there is an element y_n in M such that

$$d(y_n, b) > n$$

Now consider the sequence $\{y_n\}$. This sequence can not have a convergent sub-sequence, but this contradicts the compactness of M. Hence M is bounded.

Example # Consider the sequence $\{e_n\}$ in l^2-space where

$$e_n = \delta_{nj} = (0, 0, \dots \underset{n=j}{1}, 0, 0 \dots)$$

$$\delta_{nj} = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{if } n \neq j \end{cases}$$

Clearly $\|e_n\| = 1$

$\Rightarrow$ the sequence $\{e_n\}$ is bounded. Its terms constitutes a "point set" which is closed since it has no accumulation point. Hence the point set is not a compact set

Theorem # In a finite dimensional normed space X any subset M in X is compact iff M is closed and bounded.

Proof # Let M be compact. Then from the last lemma we know that a compact subset M of a metric space is closed and bounded. Therefore M is closed and bounded.

Let us suppose M is compact and $\{x_n\}$ be a sequence in M.

Then $\{x_n\}$ will have Convergent Subsequence. Let $\{x_{n_i}\}$ be cgt Subsequence which converges to a point $x \in M$.

Then $$\|x_n - x\| = \lim_{n\to\infty} \|x_{n_i} - x\|$$
$$= 0$$
$$\Rightarrow x_n \to x \in M \quad \text{as } n \to \infty$$
i.e M is closed.

Also
$$\big|\,|x_n| - |x|\,\big| \le \|x_n - x\| < \epsilon$$
i.e every sequence $\{x_n\}$ in M is bounded.

Conversely Suppose that M is closed and bounded. we show that M is Compact.

Since X is finite dimensional, Dim $= n$

Let $\{e_1, e_2, \dots e_n\}$ be a basis for X. Now Consider any sequence $\{x_n\}$ in M. Then

$$x_m = x_1^m e_1 + x_2^m e_2 + \dots x_n^m e_n$$
$$x_i^m \in R \text{ or } C$$

As M is bounded so $\{x_n\}$ is also bounded i.e
$$\|x_m\| \le k \qquad m = 1, 2, \dots \rightarrow (1)$$
$\because \{e_1, e_2, \dots e_n\}$ is linearly independent

$\therefore \exists$ a +ve real number c such that
$$\|x_1^m e_1 + x_2^m e_2 + \dots x_n^m e_n\| \ge c(|x_1^m| + |x_2^m| + \dots |x_n^m|)$$
$$\|x_m\| \ge c(|x_1^m| + |x_2^m| + \dots + |x_n^m|)$$
$$k \ge \|x_m\| \ge c \sum_{i=1}^{n} |x_i^m|$$
$$\Rightarrow \sum_{i=1}^{n} |x_i^m| \le \|x_m\| \le k/c$$

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$\Rightarrow |x_i^m| < k/c$

$\Rightarrow \{x_i^m\}$ is bounded sequence of real numbers and will have a accumulation point x_i $\quad i = 1, 2, \cdots$

$$\left\{ \text{Let } x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \right\}$$

By Bolzano - Weierstrass theorem, this sequence has a convergent subsequence $\{x_i^{(m_j)}\}$. Suppose that the subsequence converges to x_i

Let $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

Since M is closed, $x \in M$. But then

$$\{x_i^{(m_j)}\}$$

is a subsequence $\{x^{(m)}\}$ which converges to $x \in M$. Hence M is compact.

Problem # Show that the closure $\bar{Y}$ of a subspace Y of a normed space X is again a vector space

Sol# Let $y_1, y_2 \in \bar{Y}$

$\Rightarrow$ There is a sequence $\{y_n\}$ in Y such that

$$y_n \longrightarrow y_1$$

and there is a sequence $\{x_n\}$ in Y such that

$$x_n \longrightarrow y_2$$

Since Y is a subspace of a normed space X, $\{y_n + x_n\}$ is a sequence in Y such that

$y_n + x_n \longrightarrow y_1 + y_2$

$\Rightarrow y_1 + y_2 \in \bar{Y}$

Hence $\bar{Y}$ is closed under addition

Theorem # (Riez's Lemma)

Let Y be a proper closed subspace of a Normed space X. Then for any real number $\theta \in (0\ 1)$ is a $x_\theta \in X$ such that

$$\|x_\theta\| = 1 \qquad \|x_\theta - y\| \geq \theta \qquad \forall\, y \in Y$$

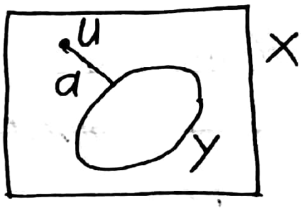
Proof # Since Y is a proper subspace of X there is

$$u \in X - Y \qquad u \notin Y$$

and let

$$a = \inf_{y \in Y} \|u - y\|$$

i.e a is a distance of u from Y



Since Y is closed, $a > 0$ for otherwise $u \in \overline{Y} = Y$ which is a contradiction because $u \notin Y$

From definition of infimum, there is a $y_0 \in Y$ such that

$$a \leq \|u - y_0\| < a + \epsilon \qquad \forall\, \epsilon > 0$$

Taking $a + \epsilon = \frac{a}{\theta}$ $\qquad \theta \in (0\ 1)$

i.e $\epsilon = \frac{a}{\theta} - a$ $\qquad a < \frac{a}{\theta}$

Then $a \leq \|u - y_0\| \leq \frac{a}{\theta}$

Now put

$$x_\theta = c(u - y_0)$$

where $c = \dfrac{1}{\|u - y_0\|}$

$\because \|u - y_0\| \leq \frac{a}{\theta}$

$c = \frac{1}{\|u - y_0\|} \geq \frac{\theta}{a}$

$$\|x_\theta\| = \|c(u - y_0)\|$$

$$= \frac{\|u - y_0\|}{\|u - y_0\|} = 1$$

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$$\|x_\theta\| = 1$$

Also $\|x_\theta - y\| = \|c(u - y_0) - y\|$

$$= c\,\| u - y_0 - \tfrac{1}{c}\cdot y \|$$
$$= c\,\| u - (y_0 + y/c) \|$$
$$= c\,\| u - y_1 \| \qquad y_1 = y_0 + \tfrac{y}{c} \in Y$$

By definition of vector space

So
$$\| x_0 - y \| = c\,\| u - y_1 \| \geq c a \qquad \because \| u - y \| \geq a \;\; \forall y \in Y$$
$$\geq \frac{1}{\| u - y_0 \|} \cdot a$$

Because (a) is $\inf_{y \in Y} \| u - y \|$

$$\geq a \cdot \theta/a = \theta$$

Hence $\| x_0 - y \| \geq \theta \quad \forall y \in Y$ ($\because y$ is arbitrary)

This Completes the proof:

Theorm on Applications of Rieze,s Lemma

Theorem# Let N be a normed linear space such that the closed unit ball

$$\bar{B}(0,1) = \{ x \in N : \| x \| \leq 1 \} \text{ is Compact.}$$

Then N is finite dimensional space.

Proof# Suppose that for a normed space N, the closed unit ball $\bar{B}(0,1) = \{ x \in N : \|x\| \leq 1 \}$ is compact. Let N be infinite dimensional. Let $0 \neq x_1 \in N$. Then

$$e_1 = \frac{x_1}{\| x_1 \|} \quad \text{has unit norm}$$

So $e_1 \in \bar{B}(0,1)$

Let $N_1 = \langle e_1 \rangle$, then N_1 is one dimensional Subspace of N. Since N is infinite dimensional, N_1 is proper closed subspace of N. So that By Riez,s Lemma, there is an element $e_2 \in N - N_1$

such that

$\|e_2\| = 1$ & $\|e_2 - e_1\| \geq \theta = 1/2$ $\quad \theta = 1/2 \in (0,1)$

Take $N_2 = \langle e_1, e_2 \rangle$, then N_2 is a two dimensional space

Since N is infinite dimensional, N_2 is a proper closed subspace of N

Again By Riez,s Lemma there is an element $e_3 \in N \setminus N_2$ such that

$$\|e_3\| = 1 \qquad \|e_3 - y\| \geq \theta = 1/2 \quad \forall y \in N_2$$

In particular

$\|e_3 - e_2\| \geq 1/2$ $\qquad \|e_3 - e_2\| \geq 1/2$

Continuing in this way, we have a set of elements $e_1, e_2 - - - e_n, - - -$ such that

$\|e_i\| = 1$ & $\|e_m - e_n\| \geq \frac{1}{2}$ $\quad \forall m, n$

Because N is infinite dimensional. But all $e_1, e_2 - - - - e_n - - - \in \bar{B}(0,1)$

i.e $\{e_i\}$ is a sequence in $\bar{B}(0,1)$ which does not have any convergent subsequence because for any subsequence $\{e_{i_k}\}$

$$\|e_{i_k} - e_{i_{k'}}\| \geq 1/2$$

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Then $\{e_{i_k}\}$ is not convergent

This is a contradiction to our assumption that $\bar{B}(0,1)$ is compact. Therefore N is finite dimensional.

<u>Theorem</u># Let M be a compact subset of a normed space N. Let $f: N \longrightarrow N'$ (N' is also a normed space) be a continuous function. Then $f(M)$ is also compact.

Proof# Let $f: N \to N'$ be a continuous function. and M, a compact, subset of N. We show that $f(M)$ is compact.

Let $\{v_n\}$ be a sequence in $f(M)$. We show that this sequence has a cgt subsequence

Now

$\{u_n\} = \{f^{-1}(v_n)\}$ is a sequence in M

Since is compact, this sequence $\{u_n\}$ in M has a convergent subsequence $\{u_{n_k}\}$ which converges to $u \in M$ i.e

$$\lim_{k \to \infty} u_{n_k} = u \qquad u \in M.$$

So $f(u) \in f(M)$

By the continuity of f, $\{f(u_{n_k})\} \to \{f(u)\}$ in $f(M)$

$\Rightarrow$ $\{f(u_n)\} = \{v_n\}$ has a convergent subsequence in $f(M)$.

Thus $f(M)$ is compact

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Theorem# Let M be a compact subset of a normed space N, and $f: N \to N'$ (N' is normed space) be continuous function, then there are points a, b in M such that

$$f(a) = \sup_{x \in M} \|f(x)\| \qquad f(b) = \inf_{x \in M} \|f(x)\|$$

Proof# Since M is compact and $f: N \to N'$ is continuous with $M \subseteq N$. So $f(M)$ is compact. By a previous result, $f(M)$ as a compact subset of a metric space N is closed and bounded (i.e glb & sup exist in $f(M)$)

Thus $u = \sup_{x \in M} \|f(x)\| \qquad v = \inf_{x \in M} \|f(x)\|$

$\because f(M)$ is closed, u & v are in $f(M)$

i.e $u, v \in f(M) \Rightarrow$ there are $a, b \in M$ such that

$f(a) = \sup_{x \in M} \|f(x)\|$ & $f(b) = \inf_{x \in M} \|f(x)\|$ (Proved)